International Research Journal of Innovations in Engineering and Technology (IRJIET)

ISSN (online): 2581-3048

Volume 10, Issue 9, pp 138-149, September-2026

https://doi.org/10.47001/IRJIET/2026.109015

On the Introduction of an Upwind Differencing into a Numerical Technique to Solve Boundary Value Problems (BVP)

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***Abstract* - In practices, mathematical modelling problems can be expressed in the form of BVPs that arise frequently and majorly in the fields of science and engineering, such as electric circuits, fluid dynamics, the motion of rockets or satellites, and other areas of engineering applications. The study of Mildly Non-Linear Boundary Value Problems (MNBVP) is gaining popularity on daily bases as a result of several real-life practical models that degenerate into MNBVP. In recent time, several authors have formulate done numerical techniques or the other to determine the solutions of Boundary Value Problems (BVP); many of these techniques have not been able to easily solve Mildly Non-Linear Boundary Value Problems (MNBVP) because of its peculiarity. The MNBVP are problems that cannot be pinned as linear nor exclusively nonlinear rather; the function and it's derivative are not all constants. This paper discusses the process of using the Upwind Differencing to reduce the BVP into tridiagonal system after which the Newton-Lieberstein method is employed to solve the mildly non-linear system so formed from the BVP. This imbedded technique requires dual independent processes that involve different Methods. The technique was employed to solve some problems with numerical results showing its flexibility, robustness, and efficiency of the technique. The results compare favourably with exact or analytical results.**

Keywords: Upwind Differencing, Mildly Non-Linear Systems, Boundary Value Problem (BVP), Initial Value Problem, Tridiagonal, Difference Equation.

I. Introduction

Boundary Value Problems (BVP), unlike initial value problems, from the physical view, are second-order differential equations where the position at a particular time, $x = x_0$, is known and one intends to determine the position at a later time, $x = x_1$, such that the notion of the given differential equation is prescribed within the time interval $x_0 < x < x_1$.

By BVP, one thinks of a differential equation whose functions and/or derivatives specified at *various* points, called the boundary values. Hence, our discussion will be narrowed down to a linear differential equation. An Ordinary Differential Equation, (ODE)with boundary value is defined as:

$$y'' = f(x, y, y')x_0 < x < x_1, \qquad (1)$$

But for second-order ODE, the boundary conditions can be any of the following mentioned in [1] Dirichlet or First Kind Boundary Condition, Regular Boundary Condition, Neumann or Second Kind Boundary Condition, and Periodic Boundary Condition respectively as:

$$y(x_0) = y_0, y(x_1) = y_1, \qquad (2)$$

$$y(x_0) = y_0,\ y'(x_1) = y_1, \qquad (3)$$

$$y'(x_0) = y_0,\ y'(x_1) = y_1', \qquad (4)$$

$$y'(x_0) = y_0' y(x_1) = y_1, \qquad (5)$$

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where $x_0 \neq x_1$andthe values of x_0, x_1, y_0, and y_1 are specified constants. As viewed by [2], finding a solution $y(x)$ of (1) that will be continuous on the interval $x_0 < x < x_1$ and that will satisfy the boundary conditions (2) through (5), with an alternative approach proposed by [3].

It has been confirmed that while Initial Value Problems (IVP) have unique solutions at most time, the same is not with is not always with BVP. Some BVPs may have unique, multiple, or no solutions depending on the origin of the problem and nature. Boundary Value Problems are fundamental in mathematical modeling and such had wide practical real-life usage in Physics, Engineering, and other sciences. These entail finding solution to differential equations with specified conditions at its boundaries. This makes BVP necessary for precisely describing real-life phenomena including heat conduction, fluid dynamics, and structural analysis.

At a grid point x_i, the three approximations to the derivatives, $y_i^{'}$, are introduced to the approximate technique namely two-point forward, two-point backward, and two-point central, respectively, as:

$$y_i^{'} = \frac{y_{i+1} - y_i}{h} \tag{6}$$

$$y_i^{'} = \frac{y_i - y_{i-1}}{h} \tag{7}$$

$$y_i^{'} = \frac{y_{i+1} - y_{i-1}}{2h} \tag{8}$$

Following the same approximate differential, it will be necessary that an additional relation, $y^{''}(x_k) = y_k^{''}$, be added to the first derivatives (6) through (8) to determine the second derivatives. To obtain this, (6) and (7) have the second-order approximation, which is the difference between (6) and (7) as:

$$y_i^{''} = \frac{y_i^{'}}{h} - \frac{y_i^{'}}{h} = \frac{\frac{y_{i+1} - y_i}{h} - \frac{y_i - y_{i-1}}{h}}{h} = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \tag{9}$$

Usually called the three-point central of the approximate technique.

While (6) through (9) intuitively have been derived, each can be derived via Taylor expansion with rigorous and extensive efforts. To approximate the second derivative requires one to use the central difference formula. This is achieved by subdividing $[x_0, x_1]$ into n equal intervals with step size $h = \frac{x_1 - x_0}{n}$ and defining the grid points $x_i = x_0 + ih$, where $i = 0, 1, 2, \dots, n$. The second derivative of $y(x)$ can then be approximated at each point in the grid.

II. Mildly Nonlinear Boundary Value Problems (MNBVP)

As opined by the authors in [4], the MNBVP equation arises mostly in problems from science, engineering, and particularly in the discretization of certain nonlinear differential equations, as it is being mentioned by [5]. There are different forms of Mildly Nonlinear Boundary Value Problems (MNBVP) depending on the model statement. However, the most widely studied variant of MNBVP is of the form (1) and (2):

$$y^{''} = f(x, y, y^{'}) \tag{10}$$

$$y(x_0) = y_0, \, y(x_1) = y_1. \tag{11}$$

To ensure that solution of (1) and (2) is unique, one assumes that

$$\frac{\partial f(x,y)}{\partial y} \geq 0, x_0 \leq x \leq x_1, \text{ and } -\infty < y < \infty. \tag{12}$$

According to [6] and [7], if (12) holds, then (1) and (2) are referred to as an MNBVP.

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For linear differential equation

$$y'' + P(x)y' + Q(x)y = R(x), \tag{13}$$

where $x \in I$ and $I \in [x_0, x_1]$. Let y_0 and y_1 be specified constants, $P(x), Q(x)$, and $R(x)$ be continuous on $[x_0,\ x_1]$. Investigating the linear BVP (13) subject to the conditions (2), it is important to note that, for numerical reasons, *a priori*, the solution of the problem (8) and (9) thus exists, and that the solution is uniquely characterized. With this, in addition to the conditions listed above,, it is assumed that, if $x_0 \le x \le x_1$,

$$Q(x) \le 0 \tag{14}$$

This is a sufficient condition to ascertain the existence and uniqueness of the solution.

If y' is assumed to be an independent variable, then

$$\frac{\partial}{\partial y}[-P(x)y' - Q(x)y + R(x)] = -Q(x) \tag{15}$$

Then,

$$Q(x) \le 0. \tag{16}$$

The solution of (1) and (2) go in the same way as prescribed with the three-point central approximation

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = f(x_i, y_i) \tag{17}$$

Substituting (17) and (8) in (13) for $i = 1, 2, 3, \dots, n-1$ gives rise to

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + P(x_i)\frac{y_{i+1} - y_{i-1}}{2h} + Q(x_i)y_i = R(x_i) \tag{18}$$

The relation formed in (17) is called the MNBVP. If $f_i(x_i)$ for all $i = 1, 2, \dots, n$, are not all constants, but rather function and the derivative, $f_i'(x_i) \ge 0$, then the system is referred to as mildly nonlinear. According to [5], all such mildly nonlinear systems have solutions, and their solutions are unique. This paper is aimed at solving this class of problems fusing two different numerical techniques.

III. Methodology

The technique to be employed herein is a fusion of an Upwind Differencing Method (UDM) and a Newton-Lieberstein (NL) Algorithm. While several approaches have been employed to find a solution to tri-diagonal systems, fewhave seemed to be used to solve mildly non-linear tri-diagonal systems; hence, the need for fused approach is investigated thus:

Step 1a: Determine the nonnegative constant M which satisfies $|P(x)| \le M$ on $a \le x \le b$. The interval is divided into n equal parts by the grid points $a = x_0, x_1, x_2, \dots, x_n = b$, but let the grid step-size, h satisfy the condition

$$Mh < 2. \tag{19}$$

Step 1b: Let $y_i = y(x_i), i = 0, 1, 2, 3, \dots, n$ and recall that $y_0 = \alpha$ and $y_n = \beta$.

Step 1c: At each interior grid point,$x_i, i = 1, 2, 3, \dots, n-1$, approximate (13) by (9) and (8) as

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + P(x_i)\frac{y_{i+1} - y_{i-1}}{2h} + Q(x_i)y_i = R(x_i) \tag{20}$$

Or, equivalently, by rearranging (20) as

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Volume 10, Issue 9, pp 138-149, September-2026

https://doi.org/10.47001/IRJIET/2026.109015

$$[2 - hP(x_i)]y_{i-1} + [-4 + 2h^2Q(x_i)]y_i + [2 + hP(x_i)]y_{i+1} = 2h^2R(x_i) \qquad (21)$$

Step 1d: Write down (21) in order, for $i = 1, 2, 3, \dots, n-1,$ and insert the known values for $y_0$and y_n. This will definitely yield a tridiagonal linear algebraic equation which satisfies the conditions afore listed.

The general form for the linear algebraic system for $n = 2, 3, 4, \dots n$ equations in the n unknowns $x_1, x_2, \dots, x_n$ is given as

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_1 = f_1(x_1) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_2 = f_2(x_2) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_3 = f_3(x_3) \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = f_n(x_n) \end{array}\right\} \qquad (22)$$

In augmented matrix form, (22) is expressed as,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x_1) \\ f_2(x_2) \\ f_3(x_3) \\ \vdots \\ f_n(x_n) \end{pmatrix} \qquad (23)$$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}, \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}, \text{ and } \boldsymbol{b} = \begin{pmatrix} f_1(x_1) \\ f_2(x_2) \\ f_3(x_3) \\ \vdots \\ f_n(x_n) \end{pmatrix} \qquad (24)$$

where $\boldsymbol{A}$ is the coefficient matrix, x is the variable vector, and the solution vector $\boldsymbol{b}$ is as given. Then, (22) can be written in a compact form as

$$\boldsymbol{Ax} = \boldsymbol{b} \qquad (25)$$

The system (23) or its equivalent (25) is called a tridiagonal system if all the entries are zero except a_{ii}, $a_{j,j+1}$, and $a_{j+1,j}$ for $i = 1, 2, 3, \dots, n$, $j = 1, 2, 3, \dots, n-1$.

Solve the linear algebraic system in Step 1d by the Newton-Lieberstein technique to yield the numerical solution so required.

The Newton-Lieberstein technique algorithm is stated in the following as:

Step 2a: Guess an initial variable value for $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots, x_n^{(0)}$ and a value for ω in the range $[0, 2]$.

Step 2b: For $k = 0, 1, 2, 3, \dots$, iterate with the relations

$$x_1^{(k+1)} = x_1^{(k)} - \omega\left(\frac{a_{11}x_1^{(k)} + a_{12}x_2^{(k)} - f_1(x_1^{(k)})}{a_{11} - f_1'(x_1^{(k)})}\right) \qquad (26)$$

$$x_2^{(k+1)} = x_2^{(k)} - \omega\left(\frac{a_{21}x_1^{(k)} + a_{22}x_2^{(k)} + a_{23}x_3^{(k)} - f_2(x_2^{(k)})}{a_{22} - f_2'(x_2^{(k)})}\right) \qquad (27)$$

$$x_3^{(k+1)} = x_3^{(k)} - \omega\left(\frac{a_{32}x_2^{(k)} + a_{33}x_3^{(k)} + a_{34}x_4^{(k)} - f_3(x_3^{(k)})}{a_{33} - f_3'(x_3^{(k)})}\right) \qquad (28)$$

$$x_{n-1}^{(k+1)} = x_{n-1}^{(k)} - \omega\left(\frac{a_{n-1,n-2}x_{n-2}^{(k)} + a_{n-1,n-1}x_{n-1}^{(k)} + a_{n-1,n}x_n^{(k)} - f_{n-1}(x_{n-1}^{(k)})}{a_{n-1,n-1} - f_{n-1}'(x_{n-1}^{(k)})}\right) \qquad (29)$$

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https://doi.org/10.47001/IRJIET/2026.109015

$$x_n^{(k+1)} = x_n^{(k)} - \omega\left(\frac{a_{n,n-1}x_{n-1}^{(k)} + a_{nn}x_n^{(k)} - f_n(x_n^{(k)})}{a_{nn} - f_n'(x_n^{(k)})}\right) \qquad (30)$$

Step 2c: In determining the convergence of the method, either of the following conditions could be used as a terminating criterion:

(i) Continue the iteration until $|x_i^{(k+1)} - x_i^{(k)}| < \epsilon, i = 1, 2, 3, \ldots, n$ for the prescribed convergence tolerance ϵ.
(ii) The method enjoys a quadratic convergence principle that makes the technique said to have converged after n iterations.
(iii) where (i) and (ii) are not satisfied, the results could be explicitly compared with results from other methods.

Step 2d: Substitute the values $x_1^{(k+1)}, x_2^{(k+1)}, x_3^{(k+1)}, \ldots, x_n^{(k+1)}$ into the original system equations to check that they are an approximate solution.

Theorem 1: If (25) is a tri-diagonal system. Let the main diagonal entries be negative while the sub-diagonal and super-diagonal entries are positive, such that

$$a_{ii} < 0, i = 1, 2, 3, \ldots, n \qquad (33)$$

$$a_{i+1,i} > 0, i = 1, 2, 3, \ldots, n-1 \qquad (34)$$

$$a_{i,i+1} > 0, i = 1, 2, 3, \ldots, n-1 \qquad (35)$$

To ensure that the tridiagonal system has a unique solution, in practice, it is important that the following conditions in (33), (34), and (35) must be met. In addition to the existing conditions in (33) to (35), entries on the main diagonal are made to dominate the matrix such that the absolute value of the main diagonal entries is greater than or it is equal to the sum of all other row entries, with strict inequality for at least one row of the coefficient matrix. Precisely, let

$$-a_{11} \geq a_{12} \qquad (36)$$

$$-a_{n,n} \geq a_{n,n-1} \qquad (37)$$

$$-a_{ii} \geq a_{i,i-1}, i = 2, 3, \ldots., n-1, \qquad (38)$$

With this strict inequality condition substantiating for any one of (36) through (38), the linear algebraic system so formed will say to have a unique solution.

The knowledge of conditions for uniqueness, as it was opined by [8] and [1] becomes very relevant before the commencement of the computation process because, for systems that do not have unique solutions, computations in such systems will lead to no right or meaningful direction. If a system has more than one solution, i.e., no unique solution, applying certain computational techniques could cause a drift from one result to another, thereby confusing.

To illustrate the MNBVP transformation procedure from the BVP via the Upwind differencing to an argument matrix form that will later be solved using an appropriate numerical method, the following test problems will be considered.

Problem 1: Consider the BVP

$$y'' + 2(2-x)y' = 2(2-x) \qquad (39)$$

$$y(0) = -1, \text{ and } y(6) = 5 \qquad (40)$$

Comparing (39) and (40) with (1), (2), and (13), $0 \leq x \leq 6$ holds for $x_0 \leq x \leq x_1$, $P(x) = 2(2-x), Q(x) = 0$, and $R(x) = 2(2-x)$.

Following the steps in the algorithm, (39) and (40) with $h = 1$ yields the system of equation

$$-4y_1 + 4y_2 = 4$$

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https://doi.org/10.47001/IRJIET/2026.109015

$2y_1 - 4y_2 + 2y_3 = 0$

$4y_2 - 4y_3 = -4$ (41)

$6y_3 - 4y_4 - 2y_5 = -8$

$8y_4 - 4y_5 = 8$

If for all $i = 1, 2, \ldots, n, f_i(x_i)$ are not all constants and the derivative, $f_i^{'}(x_i) \geq 0$, then the system is referred to as a mildly nonlinear problem satisfying the conditions in Theorem 1 as in [1]. According to [9], all such mildly nonlinear system has solutions and their solutions are said to be unique. So, before one carries out the rigorous solution exercises, it is necessary and important to find out if the solution will be unique. This actually does not satisfy the aforementioned conditions in Theorem 1 of [1]. It is found out that the solution to the problem is not unique as the conditions for uniqueness are not satisfied. The problem is that

$$Max|P(x)| = Max|2(2-x)| = 8 \leq M$$

On $0 \leq x \leq 6$,and the condition $hM < 2$ has been violated.Now, it may be that (41) has a unique solution, but one is not sure, *a priori*, that this is correct.

Without further analysis, no such conclusion can be reached. Let us then approach the problem in the following way, still keeping the grid size is $h = 1$.If the interval $0 \leq x \leq 6$ is divided into six equal parts, the grid points are x_0 , x_1 ,x_2,x_3, x_4, x_5, and x_0 as $0, 1, 2, 3, 4, 5$, and 6 respectively.

Then, at each interior grid point, the differential equation (39) is then approximated by (20) for $i = 1, 2, 3, 4$, and 5 by:

$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + 2(2 - x_i)v_i = 2(2 - x_i)$ (42)

$y_{i-1} - 2y_i + y_{i+1} + 2(2 - x_i)v_i = 2(2 - x_i)$ (43)

On carefully expressing in order each of (43) for $i = 1, 2, 3, 4$, and 5 will yield:

$-2y_1 + y_2 + 2v_1 = 3$ (44)

$y_1 - 2y_2 + y_3 = 0$ (45)

$y_2 - 2y_3 + y_4 \quad -2v_3 = -2$ (46)

$y_3 - 2y_4 + y_5 - 4v_4 = -4$ (47)

$y_4 - 2y_5 - 6v_5 = -11$ (48)

In (44) to (48), either the two-point forward or two-point backward approximations is chosen to approximate y', rather than the slightly more accurate two-point central formula.

To ensure (44) to (48) satisfy the uniqueness theorem, since the coefficient of v_1 is positive in (44), the forward difference approximation is chosen, while for (45) through (48), the backward approximation is chosen but (45) requires no approximation since v_i is not included as:

$v_1 = \frac{y_2 - y_1}{h} = y_2 - y_1$ (49)

$v_3 = \frac{y_3 - y_2}{h} = y_3 - y_2$ (50)

$v_4 = \frac{y_4 - y_3}{h} = y_4 - y_3$ (51)

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https://doi.org/10.47001/IRJIET/2026.109015

$v_5 = \frac{y_5 - y_4}{h} = y_5 - y_4$ (52)

Substituting (49), (50), (51), and (52) into (44), (46), (47), and (48) respectively and evaluating yield

$-4y_1 + 3y_2 = 3$

$y_1 - 2y_2 + y_3 = 0$

$3y_2 - 4y_3 + y_4 = -2$ (53)

$5y_3 - 6y_4 + y_5 = -4$

$7y_4 - 8y_5 = -11$

which satisfies all the uniqueness conditions on investigation, then (53) can now be solved using the Newton-Lieberstein methods as in [1] to yield the approximate solution:

$y_1 = 0, y_2 = 1, y_3 = 2, y_4 = 3$, and $y_5 = 4$ (54)

The result in (54) is then compared with the analytical result to authenticate the robustness, effectiveness, and flexibility of the fused technique.

Problem 2: Solve the Mildly Nonlinear Boundary Value Problem

$y'' = e^y$ (55)

under the conditions

$y(0) = 0, y(1) = 0$ (56)

Imperatively, the Newton-Lieberstein Method will perform better if the reduced tridiagonal system has a solution that exists and the solution is unique.

To investigate if the solution of Problem 2 will exist and determine the uniqueness of the result, the derivative of the function

$f(x, y) = e^y$ (57)

is sought. Such that

$\frac{\partial}{\partial y} f(x, y) = e^y > 0.$ (58)

Thus, the solution to the problem exists, and it is unique.

Then, to solve the problem numerically, the interval $[0, 1]$ has to be sub-divided into $n = 5$ equal parts, and $h = 0.2$ gives rise to

$$x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8, x_5 = 1.0.$$

From (37), let $y(x_i) = y_i$ such that $y_0 = 0$and $y_1 = 0$. To approximate $y_i$where $i = 1, 2, \dots, n-1$, i.e., $y_1, y_2, \dots, y_5$, the differential equation is approximated at each interior grid point by the nonlinear difference equation; introducing (9) into (36) yields

$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = e^{y_i}$ for $i = 1, 2, 3, 4$ (59)

Simplifying (59) at $h = 0.2$ further gives

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https://doi.org/10.47001/IRJIET/2026.109015

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{(0.2)^2} = e^{y_i}$$

$$25y_{i-1} - 50y_i + 25y_{i+1} = e^{y} \qquad (60)$$

For $i = 1, 2, \ldots, n - 1$in (60) and substituting the boundary conditions (56), yields the mildly nonlinear tridiagonal system as

$$\left.\begin{aligned} -50y_1 + 25y_2 &= e^{y_1} \\ 25y_1 - 50y_2 + 25y_3 &= e^{y_2} \\ 25y_2 - 50y_3 + 25y_4 &= e^{y_3} \\ y_3 - 2y_4 + y_5 &= e^{y_4} \end{aligned}\right\} \qquad (61)$$

Expressing (61) in an augmented matrix form gives as:

$$\begin{pmatrix} -50 & 25 & 0 & 0 \\ 25 & -50 & 25 & 0 \\ 0 & 25 & -25 & 25 \\ 0 & 0 & 25 & -50 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} e^{y_1} \\ e^{y_2} \\ e^{y_3} \\ e^{y_4} \end{pmatrix} \qquad (62)$$

This ends the first leg of the technique, while the second leg of the fusing algorithm begins by employing the Newton-Lieberstein's Method to solve the argument matrix (62) in which the coefficient matrix of (62), is tridiagonal.

The transformed MNBVP is now being solved following [7] thus:

Step 1: Let $\omega = 1.5$, initial guess values at $k = 0$, and $y_1^{(0)} = 0, y_2^{(0)} = 0, y_3^{(0)} = 0$, and $y_4^{(0)} = 0$.

Step 2: Update the variable values using the Newton-Lieberstein algorithm thus:

$$y_1^{(1)} = y_1^{(0)} - 1.5\frac{-50y_1^{(0)} + 25y_2^{(0)} - e^{y_1^{(0)}}}{-50 - e^{y_2^{(0)}}}$$

$$= 0 - 1.5\frac{-50(0) + 0 - 1}{-50 - 1}$$

$$= -0.02941176470588$$

$$y_2^{(1)} = y_2^{(0)} - 1.5\frac{25y_1^{(1)} - 50y_2^{(0)} + 25y_3^{(0)} - e^{y_2^{(0)}}}{-50 - e^{y_2^{(0)}}}$$

$$= 0 - 1.5\frac{-0.02941176470588(25) - 0 + 0 - 1}{-50 - 1}$$

$= -0.05103806228373$

$$y_3^{(1)} = y_3^{(0)} - 1.5\frac{25y_2^{(1)} - 50y_3^{(0)} + 25y_4^{(0)} - e^{y_3^{(0)}}}{-50 - e^{y_3^{(0)}}}$$

$$= 0 - 1.5\frac{-0.05103806228383(25) - 0 + 0 - 1}{-50 - 1}$$

$$= -0.06693975167921$$

$$y_4^{(1)} = y_4^{(0)} - 1.5\frac{25y_3^{(1)} - 50y_4^{(0)} - e^{y_4^{(0)}}}{-50 - e^{y_4^{(0)}}}$$

International Research Journal of Innovations in Engineering and Technology (IRJIET)

ISSN (online): 2581-3048

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https://doi.org/10.47001/IRJIET/2026.109015

$$= 0 - 1.5\frac{-0.06693975167921(25) - 0 + 0 - 1}{-50 - 1}$$

$= -0.07863217035236$

This process looped over and the expected result is the values of the variables at the 10th iteration. However, this is done using a program, and the summary of the results of the 10 iterations is as presented below:

Table 1

It	y_1	y_2	y_3	y_4
0	0	0	0	0
1	-0.029411765	-0.051038062	-0.066939752	-0.07863217
2	-0.052259451	-0.09161646	-0.12125812	-0.07934570
3	-0.070704971	-0.12492987	-0.11918849	-0.07746480
4	-0.086011728	-0.11803368	-0.11375534	-0.074404532
5	-0.073280542	-0.10809667	-0.10689497	-0.070883222
6	-0.072330097	-0.10731082	-0.10715201	-0.072833188
7	-0.072226689	-0.10781693	-0.10883305	-0.073095983
8	-0.072651054	-0.10911506	-0.10914261	-0.073192521
9	-0.073394697	-0.10924199	-0.10915248	-0.073151516
10	-0.073116338	-0.10898076	-0.10892482	-0.073004391

The solution of the reduced tridiagonal system as, in (62), is the solution of the MNBVP given as:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -0.073116338 \\ -0.10898076 \\ -0.10892482 \\ -0.073004391 \end{pmatrix} \qquad (63)$$

For system of equation having many variables, this technique becomes very relevant though one might have to try $\omega = 1.0, 1.3, 1.5,$ or 1.7.The technique can be used to solve BVP that are not mildly nonlinear conveniently.

IV. Observations on the Algorithm

The followings are the observations made on the algorithm as, different aspects of the algorithm need to be looked into to substantiate and validate its usage. To analysis this explicitly, Problem 1 will be used as a test case.

1. The insight into the assumption in (19) is that there is a need to examine (56). According to the positon of [8] and [10], it was observed that the tridiagonal system has

$$-4 + 2h^2Q(x_i) \qquad (65)$$

which is the main diagonal elements are the coefficients in (43) for the variable y_i, that is, -50, which is the super diagonal elements which result are the coefficientsin (43) of y_{i+1}, and the sub diagonal elements which result are the coefficients in (43) of y_{i-1}.

2. It is observed that with respect to (21), in general, the main diagonal elements will be

$$2 + hP(x_i) \qquad (66)$$

and the sub-diagonal elements will be

$$2 - hP(x_i). \qquad (67)$$

3. Ensure that the conditions for existence and uniqueness of the solution are satisfied as:

International Research Journal of Innovations in Engineering and Technology (IRJIET)

ISSN (online): 2581-3048

Volume 10, Issue 9, pp 138-149, September-2026

https://doi.org/10.47001/IRJIET/2026.109015

$$-4 + 2h^2 Q(x_i) < 0 \quad (68)$$

$$2 + 2hP(x_i) > 0 \quad (69)$$

$$2 - 2hP(x_i) > 0. \quad (70)$$

However, since (16) holds, then (68) is valid. Also, by (19),

$$|hP(x)| \leq hM < 2 \quad (71)$$

Substantiates the validity of (69) and (70).

4. The assurance that the diagonal dominance stay has to be established, i.e.,

$$|-4 + 2h^2 Q(x_i)| \geq |2 + 2hP(x_i)| \quad (72)$$

$$|-4 + 2h^2 Q(x_i)| \geq |2 - 2hP(x_i)| \quad (73)$$

And that the inequality is valid for at least one row. Though this is also valid. It can be noted that

$$|-4 + 2h^2 Q(x_i)| = 4 + 2h^2 Q(x_i) \geq 4 \quad (74)$$

5. Moreover, (69) and (70)

$$|2 + 2hP(x_i)| + |2 - 2hP(x_i)|$$

$$= 2 + hP(x_i) + 2 - hP(x_i) = 4 \quad (75)$$

So that for $i = 1, 2, 3, 4,$ and $5,$ but

$$|-4 + 2h^2 Q(x_i)| \geq |2 + 2hP(x_i)| + |2 - 2hP(x_i)|. \quad (76)$$

But by the strict inequality it will hold for the first and the last equation and, hence, all the assumptions are validated.

V. Conclusion

This research stressed solving MNBVPs that result in tridiagonal systems by using the Upwind Differencing technique and the Newton-Lieberstein method. The results demonstrated that the technique provides robust, efficient, and accurate solutions with faster convergence.

The major findings are:

- One has been able to use the Upwind Differencing to decompose the BVP into tridiagonal system of equation.
- The Newton-Lieberstein method has been used to solve mildly nonlinear systems efficiently.
- The numerical technique has showed significant improvement in terms of accuracy with each step of the iteration.
- The technique has proved to be computationally efficient on comparing to the result with the classical methods.

It enhances accuracy, stability, and efficiency, making it valuable for problems with large variables. The study contributes to the field of numerical analysis by providing a technique that balances computational efficiency with precision, ensuring that BVPs can be solved more effectively. The results from the test problems establish the stability and reliability of the method, evident in the solutions, making it an effective choice for solving MNBVPs in applied mathematics, engineering, and physics.

ABBREVIATIONS

IVP: Initial Value Problem
BVP: Boundary Value Problem
MNBVP: Mildly Nonlinear Boundary Value Problem
ODE: Ordinary Differential Equation

ISSN (online): 2581-3048
Volume 10, Issue 9, pp 138-149, September-2026
https://doi.org/10.47001/IRJIET/2026.109015

AUTHOR CONTRIBUTIONS

1. **Akinmuyise Matthew Folorunsho:** Writing the original draft, data curation, and validation of results.
2. **Adebayo Kayode James:** Conceptualization, resources, methodology, writing the review and editing.
3. **Dele-Rotimi Adejoke Olumide:** Formal analysis, methodology, and Investigation.

FUNDING

This work is not supported by any external funding.

DATA AVAILABILITY STATEMENT

The data supporting the outcome of this research work has been reported in this manuscript.

CONFLICTS OF INTEREST

The authors declare no conflicts of interest.

RESEARCH FIELD

Adetolaju Olu Sunday: Artificial Intelligence, Data Science, Operations Research, Information Management, Network Theory, Game theory.
Omowaye Kehinde Solomon: Optimization; Artificial intelligence, Operations Research, Computational Mathematics, and Applied Mathematics.
Adebayo Kayode James: Optimization, Control Theory, Mathematical Modelling, Transportation Theory.

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International Research Journal of Innovations in Engineering and Technology (IRJIET)

ISSN (online): 2581-3048

Volume 10, Issue 9, pp 138-149, September-2026

https://doi.org/10.47001/IRJIET/2026.109015

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Citation of this Article:

Adetolaju Olu Sunday, Omowaye Kehinde Solomon, & Adebayo Kayode James. (2026). On the Introduction of an Upwind Differencing into a Numerical Technique to Solve Boundary Value Problems (BVP). *International Research Journal of Innovations in Engineering and Technology - IRJIET*, 10(9), 138-149. Article DOI https://doi.org/10.47001/IRJIET/2026.109015

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